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Strain and velocity gradient theory for higher-order shear deformable beams

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Abstract

The strain and velocity gradient framework is formulated for the third-order shear deformable beam theory. A variational approach is applied to determine the governing equations together with initial and boundary conditions. Within the gradient framework, the strain energy is generalized to include strain as well as strain gradient. Furthermore, the kinetic energy is also generalized to include velocity and the velocity gradient. Such approach results in the introduction of the static and kinetic internal length scales. For dynamic analysis of beams, most of the gradient theories do not take the velocity gradient into account. The model developed in this paper, depicts the influence of the velocity gradient on the governing equations and initial and boundary conditions of the third-order shear deformable theory. Through the assumption of the velocity gradients, kinematic quantities are distinguished on the microscale and on the macroscale. Finally, Timoshenko and Euler-Bernoulli beam theories are also presented by simplifying the third-order theory.

Keywords: Dynamic analysis; Shear deformable beam; Strain gradient; Velocity gradient; Variational approach.

1. Introduction

Structures are subjected to dynamic loadings in many situations which consequently necessitate their dynamic analysis. Beams are one of the most common structural elements which are used in construction. Consequently, it is necessary to investigate the dynamic behavior of these structures.

Most materials exhibit some kind of order at one or several spatial scales between the atomic scale and that of components and structures. Standard continuum theories can only be successful in describing the overall behavior of materials if the largest of these microstructural length scales is considerably smaller than the scale of application. When the relevant macroscopic length scale approaches the largest microstructural scale in a material, microstructural effects are no longer averaged out in the macroscopic response (Eringen, 1999), thus standard continuum theories are insufficient for describing the overall behavior of materials. Recently, the extension of the classical continuum mechanics towards generalized continuum mechanics provided the opportunity to explore new phenomena in material and structural modeling. The gradient elasticity and nonlocal elasticity are popular candidates among the other generalized frameworks.

Gradient elasticity theory extends the equations of classical elasticity with additional higher-order spatial derivatives of strains, stresses and/or accelerations. Strain gradient theory successfully captures size effects. Another important aspect of the gradient theory is its capability in removing the unphysical singularity at dislocation core (Lazar and Maugin, 2005, Lazar et al., 2006) and crack tips (Mousavi et al., 2014c). In dynamic analysis, size effect mostly incorporates with the explanation of dispersive wave propagation. A complete gradient theory should include gradients of strain in the generalized strain energy as well as velocity gradients in the generalized kinetic energy. Most of recent contributions, neglect the generalization of the kinetic energy, and hold on to the classical interpretation of the kinetic energy.

Gao and Park (2007) provided a three-dimensional variational formulation of a simplified strain gradient elasticity theory. They applied this three-dimensional model to a pressurized thick-walled cylinder problem. This three-dimensional formulation can be applied for the static analysis of different structures.

Papargyri-Beskou et al. (2003a) studied the bending and stability analysis of gradient elastic Euler-Bernoulli beam. The governing equations of equilibrium were obtained by both a combination of the basic equations and a variational statement. The additional boundary conditions were obtained by both variational and weighted residual approaches. The dynamic analysis of a gradient elastic Euler-Bernoulli beam was also investigated by Papargyri-Beskou et al. (2003b). Lazopoulos and Lazopoulos (2010) studied the static bending of strain gradient elastic thin beams adopting Euler-Bernoulli principle. The dynamic analysis of a microscale Timoshenko beam model based on strain gradient elasticity theory was performed by Wang et al. (2010). The governing equations as well as initial and boundary conditions were derived by using Hamilton's principle. They considered the classical kinetic energy without velocity gradients.

Buckling and bending analysis of micro-sized beams based on the Euler-Bernoulli beam theory within the strain gradient elasticity was performed by Akgöz and Civalek (2011, 2012). The governing equations and related boundary conditions were obtained via the variational principle. They also presented the buckling analysis of functionally graded Euler-Bernoulli micro-beams based on the strain gradient theory (Akgöz and Civalek, 2013). Using the method of initial value, Artan and Toksöz (2013) analyzed the stability of gradient elastic beams. Later, Challamel and Ameur (2013) studied the lateral-torsional buckling behavior of elastic micro-structured beams within gradient elasticity approach. Static, buckling and free vibration analysis of Euler-Bernoulli beam model based on a simplified strain gradient elasticity theory is presented by Liang et al. (2014).

Ramezani (2012) considered a von Karman formulation for the non-linear dynamic analysis of Timoshenko beam model within strain gradient elasticity. Governing equations of motion and boundary conditions were derived using Hamilton's principle while the classical kinetic energy is used in this study.

The theory of shear deformable beams has been developed for the analysis of thick beams within the framework of classical elasticity (Wang et al., 2000). In line with classical elasticity, shear deformable theories are essential in generalized elasticity. Recently, Challamel (2013) applied strain gradient elasticity and Eringen's nonlocal elasticity models to beam mechanics including Euler–Bernoulli, Timoshenko and higher-order shear beam models. In this paper, within the gradient elasticity theory, the strain energy is generalized to include the strain gradients while the classical kinetic energy is implemented. The dynamic analysis of functionally graded curved shear deformable micro-beam model based on the strain gradient elasticity theory was performed by Zhang et al. (2013). The size-dependent vibration of functionally graded curved micro-beams based on the modified strain gradient elasticity theory was also investigated by Ansari et al. (2013) and the results were compared with those of degenerated beam models based on the modified couple stress and the classical theories.

The analysis of Reddy–Levinson beam model within strain gradient elasticity theory (proposed by Lam et al., 2003) was performed by Wang et al. (2014). The classical kinetic energy was used in this study to develop governing equations and boundary conditions via Hamilton's principle. A nonlinear micro-beam model based on strain gradient elasticity theory with surface energy was developed by Rajabi and Ramezani (2012). They studied the effect of geometric nonlinearity and size on the frequency of nonlinear vibration. Later, Sahmani et al. (2014) studied nonlinear free vibration analysis of functionally graded third-order shear deformable micro-beams based on the modified strain gradient elasticity theory. They also considered the classical kinetic energy once formulating the governing equations and boundary conditions via variational approach.

Experimental studies are required to validate the generalized beam theories. Recently, within the couple stress theory, Romanoff and Reddy (2014) validated the modified couple stress Timoshenko beam theory for web-core sandwich panels experimentally. On the other hand, the numerical solutions to the models within gradient theory can shed more light on the accuracy and efficiency of these models. In a recent study by Pegios et al. (2014), for the static and stability analysis, finite element stiffness matrices of a gradient elastic flexural Euler-Bernoulli beam are constructed.

In most of the studies, the gradient theory is only applied to the strain energy, while the kinetic energy is considered to be the classical one. To perform a complete consistent gradient elastic analysis of structures, the strain as well as velocity gradients are to be considered for developing the strain and kinetic energies, respectively.

In this paper, the generalized strain energy including strain and strain gradients, together with the generalized kinetic energy including velocity and velocity gradients are considered. The variational approach is employed to develop the governing equations as well as the initial and boundary conditions of a third-order shear deformable beam. This formulation can be simplified to the Euler-Bernoulli and Timoshenko beam models.

This paper is organized as follows. In section 2, the three-dimensional variational approach for the dynamic analysis of gradient elastic structures is given. The strain as well as velocity gradients are considered. Section 3 deals with the analysis of a third-order shear deformable beam. The dimension reduction is applied to the three-dimensional formulation. The Euler-Bernoulli and Timoshenko beam models are derived by simplifying the third-order shear deformable beam model. Solutions to the static and free vibration analyses of a simply supported beam are presented in section 4. Section 5 gives the conclusion.

2. Variational formulation of strain and velocity gradient theory

In strain gradient elasticity, the strain energy density function (U) of a linear elastic solid is assumed to be a quadratic function in terms of strain and first-order gradient strain (Mindlin, 1964)

$$U = U\left(\varepsilon_{ij}, \partial_k \varepsilon_{ij}\right), \qquad i, j \in \{x, y, z\} \tag{1}$$

The infinitesimal elastic strain components ε_{ij} in terms of the displacement components u_j are

$$\varepsilon_{ij} = \varepsilon_{ji} = \frac{1}{2}\left(u_{j,i} + u_{i,j}\right) \tag{2}$$

where comma denotes the partial derivative. In the generalized frameworks, the variational approach is quite popular for modeling different structures. In addition to formulation of the governing equations, this approach sheds also light on the consistent boundary conditions of the structures such as gradient beams and gradient plates (Mousavi and Paavola, 2014).

Within the framework of gradient elasticity, the Cauchy-like stress tensor components σ_{ij} and double-stress tensors components τ_{ijk} are

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}, \quad \tau_{ijk} = \frac{\partial U}{\partial \varepsilon_{ij,k}}, \qquad i, j, k \in \{x, y, z\} \tag{3}$$

where $\varepsilon_{ij,k} = \partial_k \varepsilon_{ij}$. The strain energy potential is assumed as

$$U = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{2} l_s^{\,2} C_{ijmn} \varepsilon_{mn,k} \varepsilon_{ij,k} \tag{4}$$

where l_s is static internal length scale related to the strain gradient, and C_{ijkl} are the components of the elasticity tensor, which can be expressed for an isotropic material as

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il} \right) \tag{5}$$

Here and on the following pages, summation on repeated indices is assumed. Above, λ and μ denote the Lamé constants and δ_{ij} are the components of the unit second-order tensor (i.e., Kronecker delta). Thus, as a special case of Mindlin's theory (Mindlin 1964), the strain energy potential for an isotropic material is (Altan and Aifantis, 1992, Lazar et al., 2005)

$$U = \frac{1}{2} \lambda \varepsilon_{ii} \varepsilon_{kk} + \mu \varepsilon_{ij} \varepsilon_{ij} + l_s^{\,2} \left(\frac{1}{2} \lambda \varepsilon_{ii,k} \varepsilon_{jj,k} + \mu \varepsilon_{ij,k} \varepsilon_{ij,k} \right) \tag{6}$$

This special form of gradient theory is also named as gradient elasticity theory of Helmholtz type (Lazar and Maugin, 2005, Lazar et al., 2005, Lazar, 2014)

The constitutive relations in (3) take the form

$$\begin{aligned} \sigma_{ij} &= \lambda \delta_{ij} \varepsilon_{mm} + 2\mu \varepsilon_{ij} \\ \tau_{ijk} &= l_s^{\,2} \left(\lambda \delta_{ij} \varepsilon_{mm,k} + 2\mu \varepsilon_{ij,k} \right) = l_s^{\,2} \sigma_{ij,k} \end{aligned} \tag{7}$$

Thus, the strain energy potential can be expressed as

$$U = \frac{1}{2} \varepsilon_{ij} \sigma_{ij} + \frac{1}{2} \varepsilon_{ij,k} \tau_{ijk} \tag{8}$$

Due to the symmetry of stress tensors, U takes the following form

$$U = \frac{1}{2} u_{i,j} \sigma_{ij} + \frac{1}{2} u_{i,jk} \tau_{ijk} \tag{9}$$

The strain energy U_t in a region Ω occupied by the elastically deformed material (at time t) reads

$$U_t = \int_{\Omega} U \, \mathrm{d}v = \frac{1}{2} \int_{\Omega} \left(u_{i,j} \sigma_{ij} + u_{i,jk} \tau_{ijk} \right) \mathrm{d}v. \tag{10}$$

Finally, the variation of the strain energy (U_t) is

$$\delta U_t = \int_\Omega \left(\sigma_{ij}\delta\varepsilon_{ij} + \tau_{ijk}\delta\varepsilon_{ij,k}\right)\mathrm{d}v = \int_\Omega \left(\sigma_{ij}\delta u_{i,j} + \tau_{ijk}\delta u_{i,jk}\right)\mathrm{d}v. \tag{11}$$

The following form is adopted for the variation of the external work (Mindlin, 1963)

$$\delta W_t = \int_\Omega f_i \delta u_i \mathrm{d}v + \int_{\partial\Omega} \left(t_i \delta u_i + q_i n_j \delta u_{i,j}\right)\mathrm{d}a, \tag{12}$$

where $\partial\Omega$ is the bounding (closed) surface of Ω, f_i are body forces and t_i and q_i are Cauchy and double stress traction vectors on the boundary, respectively.

According to Mindlin (1964) and also presented by Polizzotto (2012), in order to determine the kinetic energy in the gradient elasticity theory, kinematic quantities should be distinguished on the microscale and on the macroscale (Mousavi et al. 2014b). Mindlin (1964) suggested a generalized kinetic energy for the first gradient theory as

$$K = \frac{1}{2}\rho u_{i,t} u_{i,t} + \frac{1}{2}\rho l_k^{\,2} u_{i,jt} u_{i,jt} \tag{13}$$

Where ρ is the mass density, l_k is kinetic internal length regarding the velocity gradient, and "$(.)_{,t}$" denotes the time derivative. Thus, the kinetic energy includes the velocity gradient which is in line with strain gradient terms in strain energy density. The generalized kinetic energy density (13) can be reduced to the classical one by setting $l_k = 0$. Polizzotto (2012, 2013) showed that invariance requisite of strain energy leads to symmetry of stress as well as linear and angular momentum balance. On the other hand, the invariance requisite is not required for kinetic energy since any quantity having a dynamic significance (like the kinetic energy, the velocity and their consequences) needs only to be evaluated with respect to a Galilean reference observer, that is, one being fixed, or uniformly moving, with respect to the fixed stars.

The kinetic energy K_t in a region Ω occupied by the elastically deformed material (at time t) is

$$K_t = \int_\Omega K\,\mathrm{d}v = \frac{1}{2}\int_\Omega \rho\left(u_{i,t}u_{i,t} + l_k^{\,2} u_{i,jt}u_{i,jt}\right)\mathrm{d}v \tag{14}$$

and the variation of the kinetic energy reduces to

$$\delta K_t = \int_\Omega \rho\left(u_{i,t}\delta u_{i,t} + l_k^{\,2} u_{i,jt}\delta u_{i,jt}\right)\mathrm{d}v \tag{15}$$

According to Hamilton's principle

$$\int_0^t \left[\delta K_t - \delta U_t + \delta W_t\right]\mathrm{d}t = 0, \tag{16}$$

while the variations of the strain energy, external work and kinetic energy are given in (11, 12, 15), respectively. Application of this principle and the fundamental lemma of the calculus of variation will result in governing (motion) partial differential equations and boundary conditions in three dimensional forms. For specific structures such as beams, as the case in classical elasticity, dimension reduction can be applied to simplify the three dimensional formulation. This is the subject of the following section.

3. Reddy third-order beam theory within strain and velocity gradient theory

A beam with a rectangular cross-section of height h and width b is considered (Fig. 1). The beam is made of homogenous and isotropic material and is subjected to a lateral load $t_y(x)$ on its upper surface.

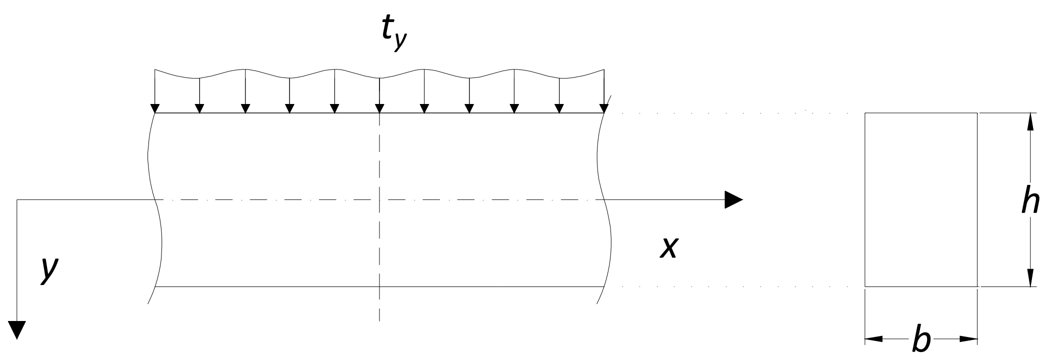

Fig. 1 Beam with rectangular cross-section subjected to lateral load $t_y(x)$

According to the Reddy beam theory (Levinson, 1981, Bickford, 1982, Reddy, 1984), the displacement field of the beam is assumed as

$$\begin{aligned} u_x(x,y,t) &= y\beta(x,t) - \alpha y^3 (\beta + w_{,x})(x,t), \\ u_y(x,y,t) &= w(x,t), \end{aligned} \tag{17}$$

where t represents time, α is a constant depending on the beam height,

$$\alpha = \frac{4}{3h^2}, \tag{18}$$

and u_x and u_y denote the displacements along the coordinates x and y, respectively. In equation (17), w represents the deflection of a point on the mid-plane and β denotes the rotation of the beam cross section. According to (2) and (17), the only nonzero components of the strain tensor are

$$\varepsilon_{xx} = y\beta_{,x} - \alpha y^3 (\beta_{,x} + w_{,xx}), \ \varepsilon_{xy} = \frac{1}{2}(1 - 3\alpha y^2)(w_{,x} + \beta), \tag{19}$$

Moreover, using (7), the nonzero components of the Cauchy and higher order stress tensors are

$$\begin{cases} \sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} \\ \sigma_{yy} = \sigma_{zz} = \lambda\varepsilon_{xx} \\ \sigma_{xy} = 2\mu\varepsilon_{xy} \end{cases}, \begin{cases} \tau_{xxx} = l_s^2 \sigma_{xx,x}, \ \tau_{xxy} = l_s^2 \sigma_{xx,y} \\ \tau_{yyy} = \tau_{zzy} = l_s^2 \sigma_{yy,y}, \ \tau_{yyx} = \tau_{zzx} = l_s^2 \sigma_{yy,x} \\ \tau_{xyx} = \tau_{yxx} = l_s^2 \sigma_{xy,x}, \ \tau_{xyy} = \tau_{yxy} = l_s^2 \sigma_{xy,y}. \end{cases} \tag{20}$$

Substituting (19) and (20) into the first variation of the strain energy (11) gives

$$
\begin{aligned}
\delta U_t = \int_\Omega \Big\{ & \sigma_{xx}\left[\left(y-\alpha y^3\right)\delta\beta_{,x} - \alpha y^3 \delta w_{,xx}\right] + \sigma_{xy}\left(1-3\alpha y^2\right)\left(\delta\beta + \delta w_{,x}\right) \\
& + l_s^2 \sigma_{xx,x}\left[\left(y-\alpha y^3\right)\delta\beta_{,xx} - \alpha y^3 \delta w_{,xxx}\right] + l_s^2 \sigma_{xy,x}\left(1-3\alpha y^2\right)\left(\delta\beta_{,x} + \delta w_{,xx}\right) \\
& + l_s^2 \sigma_{xx,y}\left[\left(1-3\alpha y^2\right)\delta\beta_{,x} - 3\alpha y^2 \delta w_{,xx}\right] - l_s^2 \sigma_{xy,y}\left[6\alpha y\left(\delta\beta + \delta w_{,x}\right)\right]\Big\} \mathrm{d}v.
\end{aligned} \tag{21}
$$

In order to apply the dimension reduction, the general bending moments and shear forces are defined as

$$
\begin{gathered}
M_{xx} := \int_A \sigma_{xx} y \mathrm{d}A,\ P_{xx} := \int_A \sigma_{xx} y^3 \mathrm{d}A,\ Q_{xx} := \int_A \sigma_{xy} \mathrm{d}A,\ R_{xx} := \int_A \sigma_{xy} y^2 \mathrm{d}A, \\
\overline{N}_{xx} := \int_A \sigma_{xx,y} \mathrm{d}A,\ \overline{S}_{xx} := \int_A \sigma_{xx,y} y^2 \mathrm{d}A,\ \overline{T}_{xx} := \int_A \sigma_{xy,y} y \mathrm{d}A,
\end{gathered} \tag{22}
$$

where A represents the cross-section area of the beam. In a beam with a large aspect ratio, the Poisson effect may be neglected to simplify the beam theory (Dym and Shames, 2013). Using equations (20) for stress components and setting Poisson's ratio $v = 0$, the general bending moments and shear force in terms of deflection can be written as

$$
\begin{aligned}
& M_{xx} = \hat{D}_{xx}\beta_{,x} - \alpha F_{xx} w_{,xx},\ P_{xx} = \hat{F}_{xx}\beta_{,x} - \alpha H_{xx} w_{,xx},\ Q_{xx} = \hat{A}_{xy}\left(\beta + w_{,x}\right),\ R_{xx} = \hat{D}_{xy} \\
& \overline{N}_{xx} = \hat{A}_{xx}\beta_{,x} - 3\alpha D_{xx} w_{,xx},\ \overline{S}_{xx} = \hat{D}_{xx}\beta_{,x} - 3\alpha F_{xx} w_{,xx},\ \overline{T}_{xx} = -6\alpha D_{xy}\left(\beta + w_{,x}\right)
\end{aligned} \tag{23}
$$

where

$$
\begin{aligned}
\left(A_{xx}, D_{xx}, F_{xx}, H_{xx}\right) &= \int_A E\left(1, y^2, y^4, y^6\right) \mathrm{d}A, \\
\left(A_{xy}, D_{xy}, F_{xy}\right) &= \int_A G\left(1, y^2, y^4\right) \mathrm{d}A
\end{aligned} \tag{24}
$$

and

$$
\begin{gathered}
\hat{D}_{xx} = D_{xx} - \alpha F_{xx},\ \hat{F}_{xx} = F_{xx} - \alpha H_{xx},\ \hat{A}_{xy} = A_{xy} - 3\alpha D_{xy}, \\
\hat{A}_{xx} = A_{xx} - 3\alpha D_{xx},\ \hat{D}_{xy} = D_{xy} - 3\alpha F_{xy}.
\end{gathered} \tag{25}
$$

In equations (24), E and G denote the elastic and shear moduli, respectively. Using definitions of the bending moments and shear forces (22), one can write the variation of the strain energy for shear deformable beam as

$$\begin{aligned}\delta U_t = \int_0^L \Big\{ & M_{xx}\delta\beta_{,x} - \alpha P_{xx}\left(\delta\beta_{,x} + \delta w_{,xx}\right) + \left(Q_{xx} - 3\alpha R_{xx}\right)\left(\delta\beta + \delta w_{,x}\right) \\ & + l_s^2 M_{xx,x}\delta\beta_{,xx} - l_s^2\alpha P_{xx,x}\left(\delta\beta_{,xx} + \delta w_{,xxx}\right) + l_s^2\left(Q_{xx,x} - 3\alpha R_{xx,x}\right)\left(\delta\beta_{,x} + \delta w_{,xx}\right) \\ & + l_s^2 \overline{N}_{xx}\delta\beta_{,x} - 3l_s^2\alpha\overline{S}_{xx}\left(\delta\beta_{,x} + \delta w_{,xx}\right) - 6l_s^2\alpha\overline{T}_{xx}\left(\delta\beta + \delta w_{,x}\right)\Big\}\,\mathrm{d}x,\end{aligned} \tag{26}$$

where L is the length of the beam. Applying Green's theorem to equation (26) results in

$$\begin{aligned}\delta U_t = & \int_0^L \left[-\hat{M}_{xx,x} + \hat{Q}_{xx} + l_s^2\left(\hat{M}_{xx,xxx} - \hat{Q}_{xx,xx} - \hat{N}_{xx,x} - 6\alpha\overline{T}_{xx}\right)\right]\delta\beta\,\mathrm{d}x \\ & + \int_0^L \left[-\alpha P_{xx,xx} - \hat{Q}_{xx,x} + l_s^2\left(\alpha P_{xx,xxxx} + \hat{Q}_{xx,xxx} - 3\alpha\overline{S}_{xx,xx} + 6\alpha\overline{T}_{xx,x}\right)\right]\delta w\,\mathrm{d}x \\ & + \left[\hat{M}_{xx} + l_s^2\left(-\hat{M}_{xx,xx} + \hat{Q}_{xx,x} + \hat{N}_{xx}\right)\right]\delta\beta\Big|_0^L + l_s^2\hat{M}_{xx,x}\beta_{,x}\Big|_0^L \\ & + \left[\alpha P_{xx,x} + \hat{Q}_{xx} + l_s^2\left(-\alpha P_{xx,xxx} - \hat{Q}_{xx,xx} + 3\alpha\overline{S}_{xx,x} - 6\alpha\overline{T}_{xx}\right)\right]\delta w\Big|_0^L \\ & + \left[-\alpha P_{xx} + l_s^2\left(\alpha P_{xx,xx} + \hat{Q}_{xx,x} - 3\alpha\overline{S}_{xx}\right)\right]\delta w_{,x}\Big|_0^L - l_s^2\alpha P_{xx,x}\delta w_{,xx}\Big|_0^L,\end{aligned} \tag{27}$$

where

$$\hat{M}_{xx} = M_{xx} - \alpha P_{xx},\ \hat{Q}_{xx} = Q_{xx} - 3\alpha R_{xx},\ \hat{N}_{xx} = \overline{N}_{xx} - 3\alpha\overline{S}_{xx}. \tag{28}$$

Moreover, substitution of equation (17) into the variation of the kinetic energy (15) results in

$$\begin{aligned}\delta K_t = \rho\int_0^L\int_A \Big\{ & \left(y^2 + \alpha^2 y^6 - 2\alpha y^4\right)\beta_{,t}\delta\beta_{,t} + \left(\alpha^2 y^6 - \alpha y^4\right)w_{,xt}\delta\beta_{,t} \\ & + \alpha^2 y^6 w_{,xt}\delta w_{,xt} + \left(\alpha^2 y^6 - \alpha y^4\right)\beta_{,t}\delta w_{,xt} + w_{,t}\delta w_{,t}\Big\}\,\mathrm{d}A\mathrm{d}x \\ + \rho l_k^2\int_0^L\int_A \Big\{ & \left(1 + 9\alpha^2 y^4 - 6\alpha y^2\right)\beta_{,t}\delta\beta_{,t} + \left(1 + 9\alpha^2 y^4\right)w_{,xt}\delta w_{,xt} \\ & + \left(y^2 + \alpha^2 y^6 - 2\alpha y^4\right)\beta_{,xt}\delta\beta_{,xt} + \left(2\alpha^2 y^6 - 2\alpha y^4\right)\beta_{,xt}\delta w_{,xxt} \\ & + \left(2\alpha^2 y^6 - 2\alpha y^4\right)\delta\beta_{,xt}w_{,xxt} + \alpha^2 y^6 w_{,xxt}\delta w_{,xxt} \\ & + \left(18\alpha^2 y^4 - 6\alpha y^2\right)\beta_{,t}\delta w_{,xt} + \left(18\alpha^2 y^4 - 6\alpha y^2\right)w_{,xt}\delta\beta_{,t}\Big\}\,\mathrm{d}A\mathrm{d}x.\end{aligned} \tag{29}$$

Integrating equation (29) over the time domain $(0,t)$ and applying Green's theorem leads to

$$
\begin{aligned}
\int_0^t \delta K_t dt &= \rho \int_0^t \int_A \int_0^L \Big\{ \Big[-\left(y^2+\alpha^2 y^6-2\alpha y^4\right)\beta_{,tt} - \left(\alpha^2 y^6-\alpha y^4\right) w_{,xtt} \Big] \delta\beta \\
&+ \Big[\alpha^2 y^6 w_{,xxtt} + \left(\alpha^2 y^6-\alpha y^4\right)\beta_{,xtt} - w_{,tt} \Big] \delta w \Big\} \mathrm{d}x\mathrm{d}A\mathrm{d}t \\
&+ \rho l_k^2 \int_0^t \int_A \int_0^L \Big\{ \Big[-\left(1+9\alpha^2 y^4-6\alpha y^2\right)\beta_{,tt} - \left(9\alpha^2 y^4-3\alpha y^2\right) w_{,xtt} \\
&+ \left(y^2+\alpha^2 y^6-2\alpha y^4\right)\beta_{,xxtt} + \left(\alpha^2 y^6-\alpha y^4\right) w_{,xxxtt} \Big] \delta\beta \\
&+ \Big[-\alpha^2 y^6 w_{,xxxxtt} + \left(9\alpha^2 y^4-3\alpha y^2\right)\beta_{,xtt} \\
&+ \left(1+9\alpha^2 y^4\right) w_{,xxtt} - \left(\alpha^2 y^6-\alpha y^4\right)\beta_{,xxxtt} \Big] \delta w \Big\} \mathrm{d}x\mathrm{d}A\mathrm{d}t \\
&- \rho \int_0^t \int_A \Big[\alpha^2 y^6 w_{,xtt} + \left(\alpha^2 y^6-\alpha y^4\right)\beta_{,tt} \Big] \delta w \Big|_0^L \mathrm{d}A\mathrm{d}t \\
&+ \rho l_k^2 \int_0^t \int_A \Big\{ \Big[-\left(y^2+\alpha^2 y^6-2\alpha y^4\right)\beta_{,xtt} - \left(\alpha^2 y^6-\alpha y^4\right) w_{,xxtt} \Big] \delta\beta \\
&+ \Big[\left(\alpha^2 y^6-\alpha y^4\right)\beta_{,xxtt} + \alpha^2 y^6 w_{,xxxtt} - \left(9\alpha^2 y^4-3\alpha y^2\right)\beta_{,tt} \\
&- \left(1+9\alpha^2 y^4\right) w_{,xtt} \Big] \delta w - \Big[\left(\alpha^2 y^6-\alpha y^4\right)\beta_{,xtt} + \alpha^2 y^6 w_{,xxtt} \Big] \delta w_{,x} \Big\} \Big|_0^L \mathrm{d}A\mathrm{d}t.
\end{aligned}
\tag{30}
$$

In obtaining equation (30), the initial conditions are set equal to zero. Assuming that the transverse load $t_y(x)$ acts on the centroidal axis of the beam, the variation of the external work takes the form

$$
\delta W = \int_0^L t_y \delta w \mathrm{d}x. \tag{31}
$$

In order to derive the equations of motion, Hamilton's principle (16) is used. Substituting (27), (30) and (31) into Hamilton's principle (16) yields

$$
\begin{aligned}
&\int_0^t \int_0^L \Big\{ -\hat{M}_{xx,x} + \hat{Q}_{xx} + l_s^2 \left(\hat{M}_{xx,xxx} - \hat{Q}_{xx,xx} - \hat{N}_{xx,x} - 6\alpha \overline{T}_{xx} \right) \\
&+ \rho\left(I+\alpha^2 H-2\alpha F\right)\beta_{,tt} + \rho\left(\alpha^2 H-\alpha F\right) w_{,xtt} \\
&+ \rho l_k^2 \left(A+9\alpha^2 F-6\alpha I\right)\beta_{,tt} + \rho l_k^2 \left(9\alpha^2 F-3\alpha I\right) w_{,xtt} \\
&- \rho l_k^2 \left(I+\alpha^2 H-2\alpha F\right)\beta_{,xxtt} - \rho l_k^2 \left(\alpha^2 H-\alpha F\right) w_{,xxxtt} \Big\} \delta\beta \mathrm{d}x\mathrm{d}t = 0,
\end{aligned}
\tag{32}
$$

$$\int_0^t\int_0^L \Big\{-\alpha P_{xx,xx}-\hat{Q}_{xx,x}+l_s^2\left(\alpha P_{xx,xxxx}+\hat{Q}_{xx,xxx}-3\alpha\overline{S}_{xx,xx}+6\alpha\overline{T}_{xx,x}\right)$$
$$-t_y-\rho\alpha^2 Hw_{,xxtt}-\rho\left(\alpha^2H-\alpha F\right)\beta_{,xtt}+\rho Aw_{,tt}$$
$$+\rho l_k^2\Big[\alpha^2Hw_{,xxxxtt}-\left(9\alpha^2F-3\alpha I\right)\beta_{,xtt}-\left(A+9\alpha^2F\right)w_{,xxtt}$$
$$+\left(\alpha^2H-\alpha F\right)\beta_{,xxxtt}\Big]\Big\}\delta w\,\mathrm{d}x\mathrm{d}t=0, \tag{33}$$

where

$$\left(I,F,H\right)=\int_A\left(y^2,y^4,y^6\right)\mathrm{d}A. \tag{34}$$

In addition to the equation (33), following conditions at both ends of the beam (i.e. $x = 0$ and $x = L$) should be satisfied.

$$\begin{cases}\begin{bmatrix}\hat{M}_{xx}+l_s^2\left(-\hat{M}_{xx,xx}+\hat{Q}_{xx,x}+\hat{N}_{xx}\right)\\ +\rho l_k^2\left(I+\alpha^2H-2\alpha F\right)\beta_{,xtt}+\rho l_k^2\left(\alpha^2H-\alpha F\right)w_{,xxtt}\end{bmatrix}\delta\beta=0\\ l_s^2\hat{M}_{xx,x}\delta\beta_{,x}=0\\ \begin{bmatrix}\alpha P_{xx,x}+\hat{Q}_{xx}+l_s^2\left(-\alpha P_{xx,xxx}-\hat{Q}_{xx,xx}+3\alpha\overline{S}_{xx,x}-6\alpha\overline{T}_{xx}\right)\\ +\rho\alpha^2Hw_{,xtt}+\rho\left(\alpha^2H-\alpha F\right)\beta_{,tt}-\rho l_k^2\left(\alpha^2H-\alpha F\right)\beta_{,xxtt}\\ -\rho l_k^2\alpha^2Hw_{,xxxtt}+\rho l_k^2\left(9\alpha^2F-3\alpha I\right)\beta_{,tt}+\rho l_k^2\left(A+9\alpha^2F\right)w_{,xtt}\end{bmatrix}\delta w=0.\\ \begin{bmatrix}-\alpha P_{xx}+l_s^2\left(\alpha P_{xx,xx}+\hat{Q}_{xx,x}-3\alpha\overline{S}_{xx}\right)\\ +\rho l_k^2\left(\alpha^2H-\alpha F\right)\beta_{,xtt}+\rho l_k^2\alpha^2Hw_{,xxtt}\end{bmatrix}\delta w_{,x}=0\\ -l_s^2\alpha P_{xx,x}\delta w_{,xx}=0\end{cases} \tag{35}$$

Due to the fundamental lemma of calculus of variation, the variational equations (32) and (33) result in the governing equations of motion

$$-\hat{M}_{xx,x}+\hat{Q}_{xx}+l_s^2\left(\hat{M}_{xx,xxx}-\hat{Q}_{xx,xx}-\hat{N}_{xx,x}-6\alpha\overline{T}_{xx}\right)+\rho\left(I+\alpha^2H-2\alpha F\right)\beta_{,tt}$$
$$+\rho\left(\alpha^2H-\alpha F\right)w_{,xtt}+\rho l_k^2\left(A+9\alpha^2F-6\alpha I\right)\beta_{,tt}+\rho l_k^2\left(9\alpha^2F-3\alpha I\right)w_{,xtt}$$
$$-\rho l_k^2\left(I+\alpha^2H-2\alpha F\right)\beta_{,xxtt}-\rho l_k^2\left(\alpha^2H-\alpha F\right)w_{,xxxtt}=0, \tag{36}$$

$$-\alpha P_{xx,xx}-\hat{Q}_{xx,x}+l_s^2\left(\alpha P_{xx,xxxx}+\hat{Q}_{xx,xxx}-3\alpha\overline{S}_{xx,xx}+6\alpha\overline{T}_{xx,x}\right)-\rho\alpha^2Hw_{,xxtt}$$
$$-\rho\left(\alpha^2H-\alpha F\right)\beta_{,xtt}+\rho Aw_{,tt}+\rho l_k^2\alpha^2Hw_{,xxxxtt}-\rho l_k^2\left(9\alpha^2F-3\alpha I\right)\beta_{,xtt}$$
$$-\rho l_k^2\left(A+9\alpha^2F\right)w_{,xxtt}+\rho l_k^2\left(\alpha^2H-\alpha F\right)\beta_{,xxxtt}-t_y=0. \tag{37}$$

Taking into account the definitions of moments and shear forces (23) and (28), the total differential orders of the governing differential equation (36) in terms of rotation β and (37) in terms of displacement w, are four and six, respectively. Therefore, two boundary conditions in terms of β and three boundary conditions in terms of w are expected at the boundaries (at each end of the beam). According to equation (35), these conditions are

$$
\begin{aligned}
&\left\{\begin{array}{l}
\hat{M}_{xx}+l_s^2\left(-\hat{M}_{xx,xx}+\hat{Q}_{xx,x}+\hat{N}_{xx}\right)+\rho l_k^2\left(I+\alpha^2 H-2\alpha F\right)\beta_{,xtt} \\
+\rho l_k^2\left(\alpha^2 H-\alpha F\right)w_{,xxtt}=0
\end{array}\right\}\text{or}\quad \delta\beta=0 \\
&\left\{l_s^2\hat{M}_{xx,x}=0\right\}\text{or}\quad \delta\beta_{,x}=0 \\
&\left\{\begin{array}{l}
\alpha P_{xx,x}+\hat{Q}_{xx}+l_s^2\left(-\alpha P_{xx,xxx}-\hat{Q}_{xx,xx}+3\alpha\bar{S}_{xx,x}-6\alpha\bar{T}_{xx}\right) \\
+\rho\alpha^2 Hw_{,xtt}+\rho\left(\alpha^2 H-\alpha F\right)\beta_{,tt}-\rho l_k^2\left(\alpha^2 H-\alpha F\right)\beta_{,xxtt} \\
-\rho l_k^2\alpha^2 Hw_{,xxxtt}+\rho l_k^2\left(9\alpha^2 F-3\alpha I\right)\beta_{,tt}+\rho l_k^2\left(A+9\alpha^2 F\right)w_{,xtt})=0
\end{array}\right\}\text{or}\quad \delta w=0 \qquad (38) \\
&\left\{\begin{array}{l}
-\alpha P_{xx}+l_s^2\left(\alpha P_{xx,xx}+\hat{Q}_{xx,x}-3\alpha\bar{S}_{xx}\right) \\
+\rho l_k^2\left(\alpha^2 H-\alpha F\right)\beta_{,xtt}+\rho l_k^2\alpha^2 Hw_{,xxtt}=0
\end{array}\right\}\text{or}\quad \delta w_{,x}=0 \\
&\left\{-l_s^2\alpha P_{xx,x}=0\right\}\text{or}\quad \delta w_{,xx}=0
\end{aligned}
$$

For a simply supported beam, the boundary conditions at $x = 0$ and $x = L$ take the form

$$
\left\{\begin{array}{l}
\hat{M}_{xx}+l_s^2\left(-\hat{M}_{xx,xx}+\hat{Q}_{xx,x}+\hat{N}_{xx}\right) \\
+\rho l_k^2\left(I+\alpha^2 H-2\alpha F\right)\beta_{,xtt}+\rho l_k^2\left(\alpha^2 H-\alpha F\right)w_{,xxtt}=0 \\
l_s^2\hat{M}_{xx,x}=0\quad\text{or}\quad\delta\beta_{,x}=0 \\
w=0 \\
-\alpha P_{xx}+l_s^2\left(\alpha P_{xx,xx}+\hat{Q}_{xx,x}-3\alpha\bar{S}_{xx}\right)+\rho l_k^2\left(\alpha^2 H-\alpha F\right)\beta_{,xtt}+\rho l_k^2\alpha^2 Hw_{,xxtt}=0 \\
-l_s^2\alpha P_{xx,x}=0\quad\text{or}\quad\delta w_{,xx}=0
\end{array}\right. \qquad (39)
$$

while for a clamped beam, we have

$$
\left\{\begin{array}{l}
\beta=0 \\
l_s^2\hat{M}_{xx,x}=0\quad\text{or}\quad\delta\beta_{,x}=0 \\
w=0 \\
-\alpha P_{xx}+l_s^2\left(\alpha P_{xx,xx}+\hat{Q}_{xx,x}-3\alpha\bar{S}_{xx}\right)+\rho l_k^2\left(\alpha^2 H-\alpha F\right)\beta_{,xtt}+\rho l_k^2\alpha^2 Hw_{,xxtt}=0 \\
-l_s^2\alpha P_{xx,x}=0\quad\text{or}\quad\delta w_{,xx}=0
\end{array}\right. \qquad (40)
$$

For the nonclassical boundary conditions (second and fifth equations in 39 and 40), the variational approach suggests two options. The selection of the nonclassical boundary conditions needs additional study on the behavior of the structure. The selection of one of the two options as the nonclassical boundary conditions is of great influence in the applicability of the numerical schemes (Mousavi et al., 2014a, Askes and Aifantis, 2011).

By setting internal static and kinetic length scales (l_s and l_k) equal to zero, the classical boundary conditions for Reddy beam are achieved.

Using (23), (28), (36) and (37), the governing motion equations can be written in terms of deflections as

$$\begin{aligned}
&-\left(\overline{D}_{xx}\beta_{,x}-\alpha\hat{F}_{xx}w_{,xx}\right)_{,x}+\overline{A}_{xy}\left(w_{,x}+\beta\right)\\
&+l_s^2\left(\overline{D}_{xx}\beta_{,x}-\alpha\hat{F}_{xx}w_{,xx}\right)_{,xxx}-l_s^2\left(\overline{A}_{xy}\left(w_{,x}+\beta\right)\right)_{,xx}\\
&-l_s^2\left(\overline{A}_{xx}\beta_{,x}+3\alpha\left(3\alpha F_{xx}-D_{xx}\right)w_{,xx}\right)_{,x}+36\alpha^2 l_s^2 D_{xy}\left(w_{,x}+\beta\right)\\
&=-\rho\left(I+\alpha^2 H-2\alpha F\right)\beta_{,tt}-\rho\left(\alpha^2 H-\alpha F\right)w_{,xtt}\\
&-\rho l_k^2\left(A+9\alpha^2 F-6\alpha I\right)\beta_{,tt}-\rho l_k^2\left(9\alpha^2 F-3\alpha I\right)w_{,xtt}\\
&+\rho l_k^2\left(I+\alpha^2 H-2\alpha F\right)\beta_{,xxtt}+\rho l_k^2\left(\alpha^2 H-\alpha F\right)w_{,xxxtt},
\end{aligned} \tag{41}$$

$$\begin{aligned}
&-\alpha\left(\hat{F}_{xx}\beta_{,x}-\alpha H_{xx}w_{,xx}\right)_{,xx}-\left(\overline{A}_{xy}\left(w_{,x}+\beta\right)\right)_{,x}+l_s^2\alpha\left(\hat{F}_{xx}\beta_{,x}-\alpha H_{xx}w_{,xx}\right)_{,xxxx}\\
&+l_s^2\left(\overline{A}_{xy}\left(w_{,x}+\beta\right)\right)_{,xxx}-3\alpha l_s^2\left(\hat{D}_{xx}\beta_{,x}-3\alpha F_{xx}w_{,xx}\right)_{,xx}-36\alpha^2 l_s^2\left(D_{xy}\left(w_{,x}+\beta\right)\right)_{,x}-t_y\\
&=\rho\alpha^2 H w_{,xxtt}+\rho\left(\alpha^2 H-\alpha F\right)\beta_{,xtt}-\rho A w_{,tt}-\rho l_k^2\alpha^2 H w_{,xxxxtt}\\
&+\rho l_k^2\left(9\alpha^2 F-3\alpha I\right)\beta_{,xtt}+\rho l_k^2\left(A+9\alpha^2 F\right)w_{,xxtt}-\rho l_k^2\left(\alpha^2 H-\alpha F\right)\beta_{,xxxtt}.
\end{aligned} \tag{42}$$

where

$$\overline{A}_{xy}=\hat{A}_{xy}-3\alpha\hat{D}_{xy},\overline{A}_{xx}=\hat{A}_{xx}-3\alpha\hat{D}_{xx},\overline{D}_{xx}=\hat{D}_{xx}-\alpha\hat{F}_{xx},\overline{F}_{xx}=\hat{F}_{xx}-\alpha\hat{H}_{xx}. \tag{43}$$

By setting $l_s = l_k = 0$, equations (41) and (42) reduce to

$$\begin{aligned}
&-\left(\overline{D}_{xx}\beta_{,x}-\alpha\hat{F}_{xx}w_{,xx}\right)_{,x}+\overline{A}_{xy}\left(w_{,x}+\beta\right)=\\
&\qquad -\rho\left(I+\alpha^2 H-2\alpha F\right)\beta_{,tt}-\rho\left(\alpha^2 H-\alpha F\right)w_{,xtt},
\end{aligned} \tag{44}$$

$$\begin{aligned}
&-\alpha\left(\hat{F}_{xx}\beta_{,x}-\alpha H_{xx}w_{,xx}\right)_{,xx}-\left(\overline{A}_{xy}\left(w_{,x}+\beta\right)\right)_{,x}-t_y=\\
&\qquad \rho\alpha^2 H w_{,xxtt}+\rho\left(\alpha^2 H-\alpha F\right)\beta_{,xtt}-\rho A w_{,tt}.
\end{aligned} \tag{45}$$

which are the classical motion equations of Reddy beam.

3.1. Timoshenko beam theory within strain and velocity gradient theory

The displacement field of the Timoshenko beam can be achieved by setting $\alpha = 0$ in (17) as

$$
\begin{aligned}
&u_x(x,y,t) = y\beta(x,t) \\
&u_y(x,t) = w(x,t).
\end{aligned}
\tag{46}
$$

In the same manner, the governing differential equations for Timoshenko beam can be obtained by assuming $\alpha = 0$ in (23), (36) and (37) and setting

$$
Q_{xx} := \kappa \int_A \sigma_{xy} \mathrm{d}A,
\tag{47}
$$

where κ is the shear correction factor for Timoshenko beam. Similar to the procedure for the third-order beam theory (equations 41 and 42), the motion equation of the Timoshenko beam reads

$$
\begin{aligned}
&-\left(D_{xx}\beta_{,x}\right)_{,x} + \kappa\left[A_{xy}\left(w_{,x}+\beta\right)\right] \\
&+l_s^2\left\{\left(D_{xx}\beta_{,x}\right)_{,xxx} - \kappa\left[A_{xy}\left(w_{,x}+\beta\right)\right]_{,xx} - \left(A_{xx}\beta_{,x}\right)_{,x}\right\} = \\
&-\rho I\beta_{,tt} - \rho l_k^2 A\beta_{,tt} + \rho l_k^2 I\beta_{,xxtt}
\end{aligned}
\tag{48}
$$

$$
-\kappa\left[A_{xy}\left(w_{,x}+\beta\right)\right]_{,x} + l_s^2\kappa\left[A_{xy}\left(w_{,x}+\beta\right)\right]_{,xxx} - t_y = -\rho A w_{,tt} + \rho l_k^2 A w_{,xxtt}
\tag{49}
$$

According to (39), the boundary conditions for a simply supported beam at $x = 0$ and $x = L$ are

$$
\begin{cases}
M_{xx} + l_s^2\left(-M_{xx,xx} + Q_{xx,x} + N_{xx}\right) + \rho l_k^2 I\beta_{,xtt} = 0 \\
l_s^2 M_{xx,x} = 0 \quad \text{or} \quad \delta\beta_{,x} = 0 \\
w = 0 \\
l_s^2 Q_{xx,x} = 0
\end{cases}
\tag{50}
$$

while for a clamped beam, equations (40) lead to

$$
\begin{cases}
\beta = 0 \\
l_s^2 M_{xx,x} = 0 \quad \text{or} \quad \delta\beta_{,x} \\
w = 0 \\
l_s^2 Q_{xx,x} = 0
\end{cases}
\tag{51}
$$

Clearly, when the material scale parameters l_s and l_k are set equal to zero, the classical boundary conditions for Timoshenko beam are achieved and the governing differential equations (48) and (49) reduce to following classical Timoshenko beam equations

$$
\begin{aligned}
&-\left(D_{xx}\beta_{,x}\right)_{,x}+\kappa\left(A_{xy}\left(w_{,x}+\beta\right)\right)=-\rho I\beta_{,tt},\\
&-\kappa\left(A_{xy}\left(w_{,x}+\beta\right)\right)_{,x}-t_{y}=-\rho A w_{,tt}.
\end{aligned}
\tag{52}
$$

3.2. Euler-Bernoulli beam theory within strain and velocity gradient theory

Adopting the Euler-Bernoulli beam theory, the displacement field of the beam is assumed as

$$
\begin{aligned}
&u_x\left(x,y,t\right)=-yw_{,x}\left(x,t\right)\\
&u_y\left(x,y,t\right)=w\left(x,t\right)
\end{aligned}
\tag{53}
$$

According to (7) and (53) and setting Poisson's ratio $\nu=0$, the nonzero components of the strain and stress tensors take the form

$$
\begin{aligned}
&\varepsilon_{xx}=-yw_{,xx}\\
&\sigma_{xx}=E\varepsilon_{xx},\ \sigma_{yy}=\sigma_{zz}=\lambda\varepsilon_{xx}\\
&\tau_{xxx}=l_s^2\sigma_{xx,x},\ \tau_{yyx}=\tau_{zzx}=l_s^2\sigma_{yy,x}\\
&\tau_{xxy}=l_s^2\sigma_{xx,y},\ \tau_{yyy}=\tau_{zzy}=l_s^2\sigma_{yy,y}.
\end{aligned}
\tag{54}
$$

In the framework of gradient elasticity, the variation of the strain energy for Euler-Bernoulli beam in terms of bending moment and shear force can be written as

$$
\delta U_t=-\int_0^L\left[\left(M_{xx}+l_s^2\bar{N}_{xx}\right)\delta w_{,xx}+l_s^2M_{xx,x}\delta w_{,xxx}\right]\mathrm{d}x
\tag{55}
$$

where M_{xx} and $\overline{N}_{xx}$ are defined in equation (22). Furthermore, the kinetic energy of the Euler-Bernoulli beam is given by

$$
K_t=\frac{1}{2}\rho\int_0^L\int_A\left[u_{x,t}^2+u_{y,t}^2+l_k^2\left(u_{x,xt}^2+u_{x,yt}^2+u_{y,xt}^2\right)\right]\mathrm{d}A\mathrm{d}x.
\tag{56}
$$

In classical formulation of the Euler-Bernoulli beam ($l_k=0$), the kinetic energy is composed of two parts. The first term $(u_{x,t}^2)$ represents the kinetic energy due to rotation of the beam elements while the second term $(u_{y,t}^2)$ represents the kinetic energy due to translatory motion in the vertical direction. For thin beams, the kinetic energy due to rotation may be neglected when compared to the other term (Dym and Shames, 2013). For the formulation of gradient theory, since the gradient terms of rotational kinetic energy might not be insignificant, we do not neglect the rotatory inertia and its companion in gradient theory $(l_k^2[u_{x,xt}^2+u_{x,yt}^2])$.

The first variation of the kinetic energy is

$$\delta K_t = \rho \int_0^L \int_A \left(y^2 w_{,xt} \delta w_{,xt} + w_{,t} \delta w_{,t} \right) \mathrm{d}A\mathrm{d}x + \rho l_k^2 \int_0^L \int_A \left(y^2 w_{,xxt} \delta w_{,xxt} + 2 w_{,xt} \delta w_{,xt} \right) \mathrm{d}A\mathrm{d}x. \tag{57}$$

The differential equation for an Euler-Bernoulli beam in the framework of gradient elasticity are obtained using a similar procedure described in the previous section as

$$\begin{aligned} &-M_{xx,xx} + l_s^2 M_{xx,xxxx} - l_s^2 \overline{N}_{xx,xx} + \rho A w_{,tt} - \rho I w_{,xxtt} \\ &+ \rho l_k^2 \left(I w_{,xxxxtt} - 2 A w_{,xxtt} \right) - t_y = 0, \end{aligned} \tag{58}$$

and the boundary conditions at $x = 0$ and $x = L$ reads

$$\begin{cases} M_{xx,x} - l_s^2 M_{xx,xxx} + l_s^2 \overline{N}_{xx,x} + \rho I w_{,xtt} + \rho l_k^2 \left(-I w_{,xxxtt} + 2 A w_{,xtt} \right) = 0 & \text{or} \quad \delta w = 0 \\ -M_{xx} + l_s^2 M_{xx,xx} - l_s^2 \overline{N}_{xx} + \rho l_k^2 I w_{,xxtt} = 0 & \text{or} \quad \delta w_{,x} = 0 \\ -l_s^2 M_{xx,x} = 0 & \text{or} \quad \delta w_{,xx} = 0 \end{cases} \tag{59}$$

For a simply supported beam the boundary conditions take the form

$$\begin{cases} w = 0 \\ -M_{xx} + l_s^2 M_{xx,xx} - l_s^2 \overline{N}_{xx} + \rho l_k^2 I w_{,xxtt} = 0 \\ -l_s^2 M_{xx,x} = 0 \quad \text{or} \quad \delta w_{,xx} \end{cases} \tag{60}$$

In the case of a clamped beam, the boundary conditions are

$$\begin{cases} w = 0 \\ w_{,x} = 0 \\ -l_s^2 M_{xx,x} = 0 \quad \text{or} \quad \delta w_{,xx} = 0 \end{cases} \tag{61}$$

Considering $l_s = l_k = 0$, equations (60) and (61) are reduced to the classical boundary conditions for simply supported and clamped Euler-Bernoulli beams.

The governing differential equation (58) can be written in terms of deflection as

$$\begin{aligned} &\left(D_{xx} w_{,xx} \right)_{,xx} - l_s^2 \left(D_{xx} w_{,xx} \right)_{,xxxx} + l_s^2 \left(A_{xx} w_{,xx} \right)_{,xx} - t_y \\ &= -\rho A w_{,tt} + \rho I w_{,xxtt} - \rho l_k^2 \left(I w_{,xxxxtt} - 2 A w_{,xxtt} \right) \end{aligned} \tag{62}$$

For the analysis of thin beams, having neglected the rotational kinetic energy and its gradient, the governing differential equation will be

$$\left(D_{xx} w_{,xx} \right)_{,xx} - l_s^2 \left(D_{xx} w_{,xx} \right)_{,xxxx} + l_s^2 \left(A_{xx} w_{,xx} \right)_{,xx} - t_y = -\rho A w_{,tt} + \rho l_k^2 A w_{,xxtt} \tag{63}$$

Setting $l_s = l_k = 0$, the classical differential equation for Euler-Bernoulli beam is obtained as

$$\left(D_{xx}w_{,xx}\right)_{,xx} - t_y = -\rho A w_{,tt}. \tag{64}$$

A solution for a simply supported gradient shear deformable beam is presented in the following section.

4. Analytical solution for a simply supported beam

In this section, the static and dynamic analyses of a simply supported beam are presented. The simply supported shear deformable beam is solved analytically.

4.1 Static analysis of simply supported beam

The governing differential equations (41) and (42) with the boundary conditions (39) describe the behavior of a simply supported beam subjected to load $t_y(x)$. Considering different options for the nonclassical boundary conditions, we assume the following conditions for the beam

$$\begin{cases} \hat{M}_{xx} + l_s^2\left(-\hat{M}_{xx,xx} + \hat{Q}_{xx,x} + \hat{N}_{xx}\right) \\ +\rho l_k^2\left(I + \alpha^2 H - 2\alpha F\right)\beta_{,xtt} + \rho l_k^2\left(\alpha^2 H - \alpha F\right)w_{,xxtt} = 0 \\ \delta\beta_{,x} = 0 \\ w = 0 \\ -\alpha P_{xx} + l_s^2\left(\alpha P_{xx,xx} + \hat{Q}_{xx,x} - 3\alpha\bar{S}_{xx}\right) \\ +\rho l_k^2\left(\alpha^2 H - \alpha F\right)\beta_{,xtt} + \rho l_k^2\alpha^2 H w_{,xxtt} = 0 \\ \delta w_{,xx} = 0 \end{cases} \tag{65}$$

The system of equations (41) and (42) together with conditions (65) has a solution in the form of

$$w(x) = \sum_{n=1}^{\infty} w_n \sin\left(\frac{n\pi x}{L}\right), \quad \beta(x) = \sum_{n=1}^{\infty} \beta_n \cos\left(\frac{n\pi x}{L}\right) \tag{66}$$

The load $t_y(x)$ can also be expanded in the form

$$t_y(x) = \sum_{n=1}^{\infty} t_n \sin\left(\frac{n\pi x}{L}\right) \tag{67}$$

where the coefficients t_n are

$$t_n = \frac{2}{L}\int_0^L t_y(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{68}$$

Substitution of (66) and (67) into (41) and (42), yields

$$\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}\begin{bmatrix} \beta_n \\ w_n \end{bmatrix} = \begin{bmatrix} 0 \\ t_n \end{bmatrix}; \tag{69}$$

where

$$\begin{cases} k_1 = \left(1+l_s^2\gamma^2\right)a_1 + a_2 + a_3, \; k_2 = \left(1+l_s^2\gamma^2\right)a_4 + \gamma a_3 + a_5, \\ k_3 = \left(1+l_s^2\gamma^2\right)a_4 + \gamma a_3 + a_6, \; k_4 = \left(1+l_s^2\gamma^2\right)a_7 + \gamma^2 a_3 + a_8, \end{cases} \tag{70}$$

and

$$\begin{cases} a_1 = \gamma^2 \overline{D}_{xx} + \overline{A}_{xy}, \; a_2 = l_s^2 \overline{A}_{xx}, \; a_3 = 36\alpha^2 l_s^2 D_{xy}, \; a_4 = -\alpha\gamma^3 \hat{F}_{xx} + \gamma \overline{A}_{xy} \\ a_5 = 3\alpha l_s^2 \gamma^3 (3\alpha F_{xx} - D_{xx}), \; a_6 = -3\alpha l_s^2 \gamma^3 \hat{D}_{xx}, \\ a_7 = \alpha^2\gamma^4 H_{xx} + \gamma^2 \overline{A}_{xy}, \; a_8 = 9\alpha^2 l_s^2 \gamma^4 F_{xx} \end{cases} \tag{71}$$

while $\gamma = \dfrac{n\pi}{L}$. Therefore, w_n and β_n can be easily determined as

$$\beta_n = \frac{-k_2}{k_4 k_1 - k_3 k_2} t_n; \; w_n = \frac{k_1}{k_4 k_1 - k_3 k_2} t_n. \tag{72}$$

The coefficients (72) can be substituted into (66) and the analytical solution of the simply supported beam will be obtained, which can be simplified to the classical solution by assuming $l_s = l_k = 0$ (Wang et al., 2014).

To shed more light on the effect of the internal length scale, the variations of the normalized deflection and rotation versus the normalized internal length is depicted in Fig. 2. In this example, the lateral loading is assumed to be $t_y(x) = t_1 \sin\left(\pi x / L\right)$ and the deflection and rotation of the beam is normalized with their classical companion ($l_s = 0$).

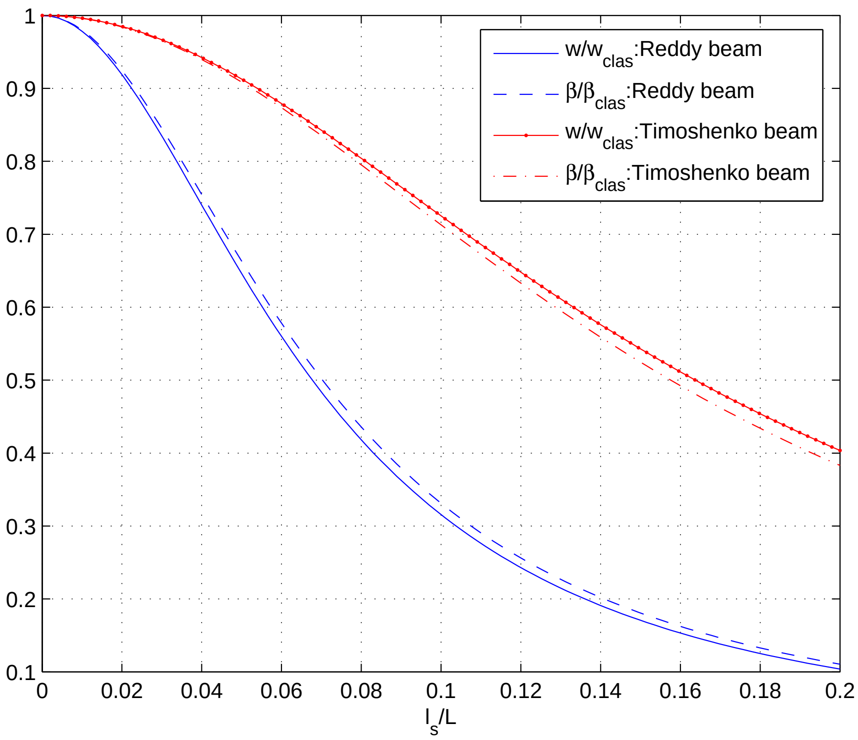

Fig. 2 Variation of the normalized central deflection and rotation of the beam under sinusoidal loading versus normalized internal length (l_s/L) for b=0.2L, h=0.2L, L=1

It is observed that by increasing the internal length, the deflection and rotation of the beam are decreasing. Furthermore, Fig. 2 depicts that for Reddy beam, the effect of the internal length is more significant than the Timoshenko beam.

4.2 Free vibration of a simply supported beam

Free vibration of the simply supported beam is governed by equations (41), (42) and the boundary conditions (65) while the load $t_y(x)$ in (42) is set equal to zero. The solution of the governing equations (41 and 42) can be assumed as

$$w(x,t)=\sum_{n=1}^{\infty} w_n^d \sin\left(\frac{n\pi x}{L}\right)e^{i\omega_n t};\beta(x,t)=\sum_{n=1}^{\infty}\beta_n^d \cos\left(\frac{n\pi x}{L}\right)e^{i\omega_n t} \tag{73}$$

which satisfies the boundary conditions (65). In the harmonic solutions (73), ω_n is the vibrational frequency and i is the imaginary number defined by i^2 = -1. Substitution of (73) into (41) and (42) leads to

$$\begin{bmatrix} k_1-k_5\omega_n^2 & k_2-k_6\omega_n^2 \\ k_3-k_6\omega_n^2 & k_4-k_7\omega_n^2 \end{bmatrix}\begin{bmatrix}\beta_n^d \\ w_n^d\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}. \tag{74}$$

where

$$\begin{cases} k_5=\left(1+l_k^2\gamma^2\right)b_1+l_k^2 b_3 \\ k_6=\left(1+l_k^2\gamma^2\right)\gamma b_2+l_k^2\gamma b_4 \\ k_7=\left(1+l_k^2\gamma^2\right)\gamma^2 b_5+b_6+l_k^2\gamma^2 b_7 \end{cases} \tag{75}$$

and

$$\begin{cases} b_1=\rho(I+\alpha^2 H-2\alpha F),\ b_2=\rho(\alpha^2 H-\alpha F),\ b_3=\rho(A+9\alpha^2 F-6\alpha I) \\ b_4=\rho(9\alpha^2 F-3\alpha I),\ \ b_5=\rho\alpha^2 H,\ b_6=\rho A,\ b_7=\rho\left(A+9\alpha^2 F\right). \end{cases} \tag{76}$$

For the existence of a non-trivial solution, the determinant of the coefficient matrix of (74) has to vanish. This condition leads to

$$R_1\omega_n^4+R_2\omega_n^2+R_3=0 \tag{77}$$

while

$$R_1=k_5k_7-k_6^2;R_2=k_3k_6+k_2k_6-k_1k_7-k_5k_4;R_3=k_1k_4-k_2k_3 \tag{78}$$

It is worth mentioning that k_1, k_2, k_3 and k_4 are functions of the static length scale (l_s), while k_5, k_6 and k_7 are functions of kinetic internal scale (l_k). Using equation (77), the solution for vibrational frequency reads

$$\omega_n = \sqrt{\frac{-R_2 + \sqrt{R_2^2 - 4R_1R_3}}{2R_1}}. \tag{79}$$

The effects of the strain gradient and velocity gradient on vibrational frequency are represented by l_s and l_k, respectively. This solution (79) can be simplified to the vibrational frequency of a simply supported Reddy beam when $l_s = l_k = 0$ (Wang at al., 2014).

Fig 3 depicts the effect of the static and kinetic length scales on the first vibrational frequency for Reddy beam. In this example, the beam is assumed to be made of epoxy with material properties E= 1.44 GPa, l_s=17.6 μm, ρ=1.22 kg/m^3.

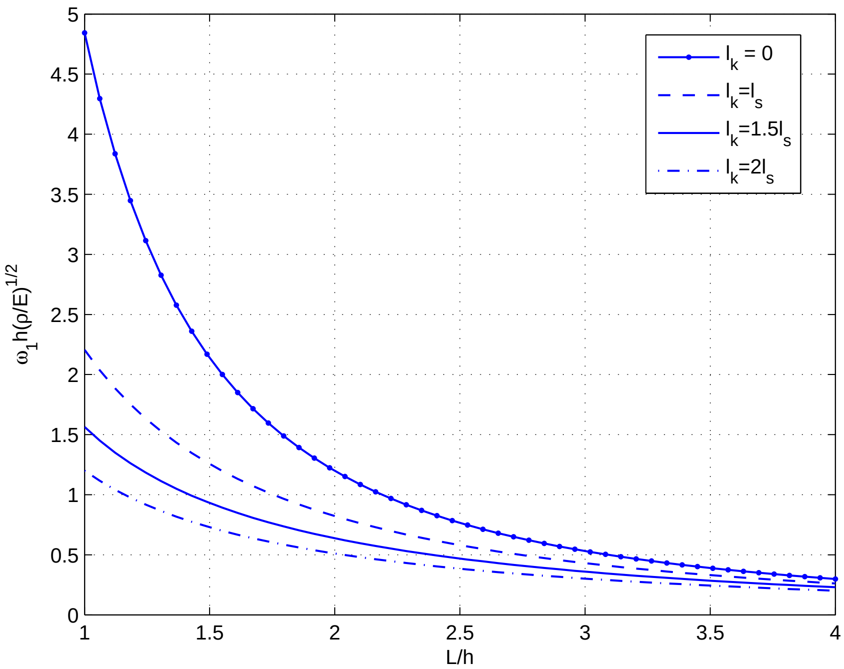

Fig. 3 Variations of the normalized first vibrational frequency of a simply supported Reddy beam versus L/h for different kinetic internal length scales, h=2l_s, b=2h

It is observed that for a beam with lower value of L/h, the velocity gradient has significant effect on the vibrational frequency. Consequently, the velocity gradient plays an important role on the dynamic behavior of micro-beam. Fig. 3 depicts that by increasing the kinetic internal length, the vibrational frequency is reduced.

5. Conclusion and summary

A dynamic third-order shear deformable beam model is formulated within velocity and strain gradient theory. The strain and kinetic energies are generalized by considering strain and velocity gradients, respectively. Through the assumption of the velocity gradient, kinematic quantities are distinguished on the microscale and on the macroscale. It is observed that the generalized kinetic energy contributes to additional terms in the governing equations as well as the initial and boundary conditions of different beam theories. Through a variational approach, the consistent classical and nonclassical boundary conditions are determined. It is observed that the variational approach gives two options for the nonclassical boundary conditions.

The current third-order shear deformable gradient model is useful for the analysis of thick micro-beams. This higher-order model is simplified to lower-order theories including Timoshenko and Euler-Bernoulli models. Analytical series solutions are provided for the static problem and free vibration of simply supported beams. The influences of the static and kinetic internal length scales on the static and dynamic behavior of the beam are depicted.

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